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ON THE COMPLEX MONGE–AMPÈRE OPERATOR IN UNBOUNDED DOMAINS

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Abstract. In this note we give sufficient conditions on a measure μ, defined on a unbounded strictly hyperconvex domain in $\mathbb{C}^n$, to be the Monge–Ampère measure of some plurisubharmonic function. These generalize recent results by Lê et al.

1. Introduction. In this note we shall generalize the main result of Cegrell's seminal article [**4**] to certain unbounded domains. For the recent progress of the complex Monge–Ampère equation on unbounded domains we refer to [**2, 11**] and the references therein.

Recall that a function $\varphi : \Omega \to (-\infty, 0)$ is called an *exhaustion function* for a connected and open set $\Omega \subseteq \mathbb{C}^n$ if the closure of the set $\{z \in \Omega : \varphi(z) < c\}$ is compact in Ω, for every $c \in (-\infty, 0)$. If Ω is bounded (or unbounded) such that we can choose φ to be bounded and plurisubharmonic, then Ω is called *hyperconvex* and if additionally φ can be chosen to be strictly plurisubharmonic, then we call Ω *strictly hyperconvex*. The assumption that Ω is (strictly) hyperconvex is a standard assumption to ensure the existence of sufficiently many plurisubharmonic functions that satisfy $\lim_{z\to\partial\Omega} \varphi(z) = 0$. The set $\mathcal{PSH}^-(\Omega)$ shall be the set of nonpositive plurisubharmonic functions defined on Ω. Following Cegrell in [**4**], we introduce the subsets $\mathcal{E}_0(\Omega)$, $\mathcal{F}(\Omega)$, and $\mathcal{E}(\Omega)$ of $\mathcal{PSH}^-(\Omega)$ (see Section 2 for details).

Our aim of this note is the following:

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THEOREM 1.1. *Let Ω be an unbounded strictly hyperconvex domain in $\mathbb{C}^n$, and let μ be a nonnegative Radon measure defined on Ω, vanishing on pluripolar sets such that there exists $H \in \mathcal{PSH}^-(\Omega) \cap L^1(\mu)$, $H \neq 0$. Then there exists $u \in \mathcal{E}(\Omega)$ such that*

$$(dd^c u)^n = \mu\,.$$

Here $(dd^c u)^n$ denotes the complex Monge–Ampère measure of u.

Theorem 1.1 generalizes Corollary 5.4 in [**11**]. We leave out the question about Cegrell classes with boundary values that was considered in [**11**]. It should be noted that it is not known wether the assumption that the domain Ω should be *strictly* hyperconvex is necessary.

As said earlier this is a generalization of Cegrell's result to unbounded strictly hyperconvex domains. An example due to Jarnicki and Zwonek [**9**] shows that there exists an unbounded hyperconvex domain in $\mathbb{C}^n$ that is not biholomorphically equivalent to any bounded pseudoconvex domain in $\mathbb{C}^n$. This shows that the complex Monge–Ampère equation on unbounded domains is considerably different from the one considered on bounded domains ([**9**], see [**2**] for details).

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2. Preliminaries. In this section we shall state some known definition and results that we shall use in the proof of our main theorem. In the original references they are stated and proved under the assumption that the underlying domain is bounded, but by following the ideas one can repeat the proofs line by line and see that the results hold for unbounded domains as well.

Following the notation introduced by Cegrell in [**4**] (see e.g. [**7**] for a detailed overview) for bounded hyperconvex domain, we define the following classes of plurisubharmonic functions on an unbounded hyperconvex domain Ω in $\mathbb{C}^n$:

$$\mathcal{E}_0(\Omega) = \left\{\varphi \in \mathcal{PSH}^- \cap L^\infty(\Omega) : \lim_{z\to\partial\Omega}\varphi(z)=0,\ \lim_{|z|\to\infty}\varphi(z)=0,\ \int_\Omega (dd^c\varphi)^n < \infty\right\},$$

$$\mathcal{F}(\Omega) = \left\{\varphi \in \mathcal{PSH}^-(\Omega) : \exists\,\{u_j\} \subset \mathcal{E}_0(\Omega),\ \varphi_j \searrow \varphi,\ \sup_j \int_\Omega (dd^c\varphi_j)^n < \infty\right\},$$

$$\mathcal{E}(\Omega) = \left\{\varphi \in \mathcal{PSH}^-(\Omega) : \forall\,\omega \Subset \Omega\ \exists\,\varphi_\omega \in \mathcal{F}(\Omega) \text{ such that } \varphi_\omega = \varphi \text{ on } \omega\right\}.$$

Note that in case of bounded hyperconvex domain Ω the condition

$$\lim_{|z|\to\infty}\varphi(z) = 0$$

in the definition of $\mathcal{E}_0(\Omega)$ is superfluous.

The following theorem is Theorem 2.1 in [**4**]:

THEOREM 2.1. *For any negative plurisubharmonic function u defined on an unbounded hyperconvex domain $\Omega \subseteq \mathbb{C}^n$ there exists a sequence $\{u_j\} \subset \mathcal{E}_0(\Omega)$ such that $u_j \searrow u$ pointwise, as $j \to \infty$.*

PROOF. We shall follow the original proof from [**4**]. Let $w \in \mathcal{E}_0(\Omega)$, $w \neq 0$. Choose an increasing sequence, Ω_j, of strictly pseudoconvex subsets of Ω such that for every $j \in \mathbb{N}$ there is $\Omega_j \Subset \Omega_{j+1}$ and

$$\bigcup_{j=1}^{\infty} \Omega_j = \Omega\,.$$

In other words, Ω_j is a fundamental sequence of Ω. Furthermore, this fundamental sequence can be chosen so that for each $j \in \mathbb{N}$ there is

$$w \geq -\frac{1}{2j^2} \qquad \text{on} \quad \Omega \backslash \Omega_j\,.$$

Let now $\{v_{k,j}\}_{k=1}^{\infty} \subset \mathcal{PSH}^-(\Omega_{j+1}) \cap C^{\infty}(\Omega_{j+1})$ be a decreasing sequence that converges pointwise to u, as $k \to \infty$, and such that $v_{j,j} \geq v_{j+1,j+1}$. Set

$$\tilde{u}_j = \begin{cases} \max\left(v_{j,j} - \frac{1}{j}\,, jw\right) & \text{on} \quad \Omega_j, \\ jw & \text{on} \quad \Omega \backslash \Omega_j\,. \end{cases}$$

Then on $\Omega_{j+1} \setminus \Omega_j$ there hold

$$jw \geq -\frac{1}{2j} > -\frac{1}{j} > v_{j,j} - \frac{1}{j}\,.$$

Hence, $\tilde{u}_j \in \mathcal{E}_0(\Omega)$ and the sequence $\tilde{u}_j$ converges pointwise to u on Ω, as $j \to \infty$. Note that the sequence $\{\tilde{u}_j\}$ is not necessarily decreasing, and therefore define

$$u_j = \sup_{k \geq j} \tilde{u}_k\,.$$

The construction of $\tilde{u}_j$ implies that

$$\tilde{u}_j + \frac{1}{j} \geq \tilde{u}_{j+1} + \frac{1}{j+1}\,,$$

which implies that for each fixed $j \in \mathbb{N}$ it follows that the sequence

$$\max\left(\tilde{u}_j, \tilde{u}_{j+1}, \ldots, \tilde{u}_{m-1}, \tilde{u}_m + \frac{1}{m}\right)$$

decreases pointwise to u_j on Ω, as $m \to \infty$. Thus, u_j is an upper semicontinuous function and we get that $u_j \in \mathcal{E}_0(\Omega)$. Finally, $\{u_j\}$ is decreasing and converges pointwise to u on Ω, as $j \to \infty$. $\square$

REMARK. If in Theorem 2.1 we assume that $\mathcal{E}_0(\Omega) \cap \mathcal{C}(\bar{\Omega}) \neq \emptyset$, then we can choose the functions in the approximating sequence to be continuous on $\bar{\Omega}$. Under the additional assumption that Ω is bounded the assumption $\mathcal{E}_0(\Omega) \cap \mathcal{C}(\bar{\Omega}) \neq \emptyset$ is superfluous (Theorem 2.1 in [**4**]).

To define the complex Monge–Ampère operator for unbounded domains we shall follow Cegrell's classical approach from [**4**] instead of subextension techniques used in [**11**]. We need a decomposition theorem guaranteeing that a smooth compactly supported function can be written as a difference of two bounded plurisubharmonic functions. This allows us to check weak*-convergence of the Monge–Ampère measures related to plurisubharmonic functions.

Using the original proof from [**4**], but with a negative and bounded strictly plurisubharmonic function ψ instead of $|z|^2$, we obtain the following.

THEOREM 2.2. *Let Ω be an unbounded hyperconvex domain for which there exists a negative, bounded strictly plurisubharmonic function ψ. Then*

$$\mathcal{C}_0^\infty(\Omega) \subset \mathcal{E}_0(\Omega) - \mathcal{E}_0(\Omega)\,.$$

PROOF. Fix $\varphi \in \mathcal{E}_0(\Omega)$. If $f \in C_0^\infty(\Omega)$, then there exists $k > 0$ such that

$$f + k\psi \in \mathcal{PSH}(\Omega)\,.$$

Now fix $a \in \mathbb{R}$, $b \geq 0$ such that

$$a < \inf_\Omega (f + k\psi) < \sup_\Omega (f + k\psi) < b$$

and define

$$u = \max(f + k\psi - b, M\varphi),$$

where $M > 0$ is chosen so that $M\varphi < \min(a - b, k\psi - b)$ on $\operatorname{supp} f$. Then $u \in \mathcal{E}_0(\Omega)$ since $u \geq M\varphi$, and $u = f + k\psi - b$ on $\operatorname{supp} f$. Observe that

$$v = \max(k\psi - b, M\varphi) \in \mathcal{E}_0(\Omega)\,,$$

and $v = k\psi - b$ on $\operatorname{supp} f$. Thus, $f = u - v$. □

By following [**4**] it can be proved that the Monge–Ampère measure of a function $u \in \mathcal{E}(\Omega)$ is well defined in the sense that for any sequence $\{u_j\} \subset \mathcal{E}_0(\Omega)$ with $u_j \searrow u$, the corresponding sequence of Monge–Ampère measures $(dd^c u_j)^n$ is weak*-convergent to some measure μ. Furthermore, the limit measure μ does not depend on the approximating sequence $\{u_j\}$.

We shall also need the following two theorems. Theorem 2.3 is basically Proposition 5.1 in [**4**].

THEOREM 2.3. *Assume that Ω is an unbounded strictly hyperconvex domain in $\mathbb{C}^n$ and let $\{u_j\} \subset \mathcal{E}(\Omega)$ be a sequence such that $u_j \searrow u$ pointwise as $j \to \infty$. Then $u \in \mathcal{E}(\Omega)$. Furthermore, if a decreasing sequence $\{u_j\} \subset \mathcal{E}_0(\Omega)$ is such that it converges pointwise to a function $u \in \mathcal{F}(\Omega)$, then for any given negative plurisubharmonic function H defined on Ω there holds*

$$\lim_{j\to\infty} \int_\Omega H(dd^c u_j)^n = \int_\Omega H(dd^c u)^n \,.$$

Theorem 2.4 was first proved in [**3**] (for a proof in our setting see e.g. Lemma 3.12 in [**7**]).

THEOREM 2.4. *Let Ω be an unbounded strictly hyperconvex domain in $\mathbb{C}^n$, w be a bounded negative plurisubharmonic function in Ω and let $u \in \mathcal{F}(\Omega)$. Then*

$$\int_\Omega (-u)^n (dd^c w)^n \leq n! \|w\|_{L^\infty}^{n-1} \int_\Omega (-w)(dd^c u)^n \,.$$

3. Proof of Theorem 1.1. We shall need the following well-known lemma that basically is contained in the proof of Theorem 4.5 in [**4**] (see also Proposition 4.5 in [**11**]). This lemma is a central tool in certain subextension techniques (see e.g. [**6, 8, 10, 12, 13**] for further information and references).

LEMMA 3.1. *Let Ω be an unbounded hyperconvex domain in $\mathbb{C}^n$ and let $\omega \Subset \Omega$ be a bounded hyperconvex domain. For any $v \in \mathcal{F}(\omega)$ we define*

$$u = \sup\{\varphi \in \mathcal{PSH}^-(\Omega) : \varphi \leq v \text{ on } \omega\} \,.$$

Then $u \in \mathcal{F}(\Omega)$ and $(dd^c u)^n \leq \chi_\omega (dd^c v)^n$.

Next, we shall give a proof of Theorem 1.1.

PROOF OF THEOREM 1.1. Assume that Ω is an unbounded strictly hyperconvex domain in $\mathbb{C}^n$, and let μ be a nonnegative Radon measure defined on Ω, vanishing on pluripolar sets and such that there exists a negative plurisubharmonic function $H \in \mathcal{PSH}^-(\Omega) \cap L^1(\mu)$, $H \neq 0$, which in particular means that

$$\int_\Omega (-H)\, d\mu < \infty \,. \tag{3.1}$$

Theorem 2.1 yields that there exists a sequence $\{H_j\} \subset \mathcal{E}_0(\Omega)$ such that $H_j \searrow H$, and then by (3.1) we have

$$\int_\Omega (-H_j)\, d\mu \leq \int_\Omega (-H)\, d\mu < \infty \,.$$

Thus, we may without loss of generality assume that $H \in \mathcal{E}_0(\Omega)$. Let $\{\Omega_j\}$ be a fundamental sequence for Ω as in the proof of Theorem 2.1, and set $\mu_j = \chi_{\Omega_j}\mu$, where χ_{Ω_j} is the characteristic function for Ω_j. Then there exists a plurisubharmonic function $v_j \in \mathcal{F}(\Omega_j)$ defined on Ω_j such that $(dd^c v_j)^n = \mu_j$ (see e.g. [**5**]), and by the comparison principle (see e.g. [**1**]) we conclude that $\{v_j\}$ is a decreasing sequence. Now define

$$u_j = \sup\{\varphi \in \mathcal{PSH}^-(\Omega) : \varphi \leq v_j \text{ on } \Omega_j\}\,.$$

Then by Lemma 3.1 we get $u_j \in \mathcal{F}(\Omega)$ and

$$(dd^c u_j)^n \leq \chi_{\Omega_j}(dd^c v_j)^n = \mu_j\,.$$

We know that v_j is decreasing, and therefore so is u_j. Using Theorem 2.4 and (3.1) together with $H \in \mathcal{E}_0(\Omega)$, it follows that

$$\begin{aligned}\int_\Omega (-u_j)^n (dd^c H)^n &\leq n!\|H\|_{L^\infty}^{n-1} \int_\Omega (-H)(dd^c u_j)^n \\ &\leq n!\|H\|_{L^\infty}^{n-1} \int_\Omega (-H)\, d\mu_j \leq n!\|H\|_{L^\infty}^{n-1} \int_\Omega (-H)\, d\mu < \infty\,.\end{aligned}$$

Thus, there exists a function u such that $u_j \searrow u$ and by Theorem 2.3 we conclude that $u \in \mathcal{E}(\Omega)$ with $(dd^c u)^n = \mu$. ☐

Corollary 3.2. *Let Ω be an unbounded strictly hyperconvex domain in $\mathbb{C}^n$. Then the set of all nonnegative, finite Radon measures defined on Ω, vanishing on pluripolar sets belongs to the range of the complex Monge–Ampère operator.*

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